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Approximation of Differentiation Operator in the Space L_2 on Semiaxis

V. V. Arestov[1*] **and M. A. Filatova**[2**]

[1] *Ural Federal University, pr. Lenina 51, Yekaterinburg, 620000 Russia; Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620990 Russia*

[2] *Ural Federal University, pr. Lenina 51, Yekaterinburg, 620000 Russia*

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Abstract—We establish an upper bound for the error of the best approximation of the first order differentiation operator by linear bounded operators on the set of twice differentiable functions in the space L_2 on the half-line. This upper bound is close to a known lower bound and improves the previously known upper bound due to E. E. Berdysheva. We use a specific operator that is introduced and studied in the paper.

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1. The problem and its historical background. In this paper we consider the problem on the best approximation of a differential operator (of the first order) by linear bounded operators on the class of twice differentiable functions in the space $L_2 = L_2(0, \infty)$ of real-valued measurable functions f, whose square is summable on the semiaxis $(0, \infty)$, which is equipped with the norm

$$\|f\| = \|f\|_{L_2(0,\infty)} = \left(\int_0^\infty |f(t)|^2 dt\right)^{1/2}.$$

More precisely, let $W_2^2 = W_2^2(0, \infty)$ be the space of functions $f \in L_2(0, \infty)$, which are defined and continuously differentiable on $[0, \infty)$, whose derivative f' is locally absolutely continuous on the semiaxis $[0, \infty)$, and the second derivative belongs to the space $L_2(0, \infty)$. In $W_2^2 = W_2^2(0, \infty)$ we extract the class $Q_2^2 = Q_2^2(0, \infty)$ of functions f such that $\|f''\| \le 1$. In what follows, $\mathscr{B} = \mathscr{B}_2(0, \infty)$ is the set of linear bounded operators in the space $L_2(0, \infty)$, and $\mathscr{B}(N)$ is the set of operators $S \in \mathscr{B}$, whose norm is bounded by the number $N > 0$, i.e., $\|S\|_{L_2 \to L_2} \le N$. For a concrete operator $S \in \mathscr{B}$ the value

$$U(S) = \sup\{\|f' - Sf\| : \ f \in Q_2^2(0, \infty)\} \tag{1}$$

is the deviation of the operator S from the differentiation operator in the space $L_2(0, \infty)$ on the class Q_2^2. The problem under considerations consists in studying the value

$$E(N) = \inf\{U(S) : S \in \mathscr{B}(N)\} \tag{2}$$

of the best approximation in the space $L_2(0, \infty)$ on the class Q_2^2 of the differentiation operator by the set $\mathscr{B}(N)$ of linear bounded operators, whose norms are bounded by the number $N > 0$.

Problem (2) is a particular case of a more general problem on the best approximation of an unbounded linear operator by linear bounded ones on some class of elements; this problem was stated in 1967 by S. B. Stechkin [1]. Many papers [2–4] are dedicated to the Stechkin problem. This problem has been most completely studied for the differentiation operator of order k on the class of n times differentiable functions in spaces $L_p(I)$, $1 \le p \le \infty$, on the numerical axis $I = (-\infty, \infty)$, and on the

*E-mail: Vitalii.Arestov@usu.ru.
**E-mail: Maria.Filatova@usu.ru.

semiaxis $I=[0,\infty)$ with $0\le k<n$. In particular, Yu. N. Subbotin and L. V. Taikov [5] have solved the latter problem in the space $L_2(-\infty,+\infty)$ for arbitrary k and n, $0<k<n$. In the space $L_2(0,\infty)$ even in the case $k=1$, $n=2$ the exact solution to this problem, i.e., the solution to problem (2), is unknown.

G. H. Hardy, J. E. Littlewood, and G. Pólya ([6], Chap. VII, §7.8) have proved that on the set $W_2^2(0,\infty)$ the following exact inequality takes place:

$$\|f'\|^2\le 2\,\|f\|\cdot\|f''\|,\quad f\in W_2^2(0,\infty). \tag{3}$$

This result gave rise to deep investigations; see papers by T. Kato [7], V. D. Everitt et al. ([8] and references therein), M. K. Kwong, A. Zettl [9], N. P. Kuptsov [10], A. P. Buslaev [11], etc. For reviews of results related to inequality (4) and close problems see [9, 3].

For $n\ge 2$ we denote by $W_2^n=W_2^n(0,\infty)$ the space of functions $f\in L_2(0,\infty)$ which are $n-1$ times continuously differentiable on the semiaxis $[0,\infty)$ and, moreover, the derivative $f^{(n-1)}$ of order $n-1$ is locally absolutely continuous on this semiaxis, while the derivative $f^{(n)}$ of order n belongs to the space $L_2(0,\infty)$. With $0<k<n$ on the set W_2^n we have the Kolmogorov inequality with a finite constant K, i.e.,

$$\|f^{(k)}\|\le K\|f\|^{\alpha}\|f^{(n)}\|^{\beta},\quad f\in W_2^n(0,\infty), \tag{4}$$
$$\alpha=\frac{n-k}{n},\quad \beta=1-\alpha=\frac{k}{n}.$$

We denote by $K=K_{k,n}$ the exact (least possible) constant in inequality (4). Result (3) means that if $k=1$ and $n=2$, then $K_{1,2}=\sqrt{2}$. For arbitrary values k and n $(0<k<n)$ the constant $K_{k,n}$ has been circumstantially investigated by N. P. Kuptsov [10]. A. P. Buslaev considered a more general problem [11], using techniques that develop ideas of methods proposed by G. H. Hardy, J. E. Littlewood, and G. Pólya ([6], Chap. VII, §7.8); in the present paper we also use these techniques.

Reasoning analogously to S. B. Stechkin [1], the exact inequality (3) implies (see, e.g., [3], §4, formula (4.6)) that value (2) allows the following lower estimate:

$$E(N)\ge\frac{1}{2N}.$$

Problem (2) was studied by A. L. Rublyev [12] and E. E. Berdysheva [13]. The inequality $E(N)\le 1/(\sqrt[3]{4}N)$ is proved in [12], and the bound

$$E(N)\le\frac{1}{\sqrt{3}N} \tag{5}$$

is obtained in [13]. Both results have been obtained with the help of concrete operators. Bound (5) is proved with the help of the operator $B:L_2\to L_2$. For the function $f\in L_2(0,\infty)$ on the semiaxis $[0,\infty)$ we consider the differential problem

$$y^{(4)}+y=f, \tag{6}$$
$$y\in L_2[0,\infty), \tag{7}$$
$$y''(0)=y'''(0)=0. \tag{8}$$

The operator B is defined by the formula

$$Bf=y',$$

where y is a solution to problem (6)–(8).

In the present paper we establish an upper estimate for the best approximation value (2), which improves estimate (5); namely, we prove the following theorem.

Theorem. *The best approximation value in problem* (2) *satisfies the inequality*

$$E(N)\le\frac{4}{7N}. \tag{9}$$

We prove this theorem with the help of an operator constructed in the following way. For a function $f \in L_2(0,\infty)$ we consider the problem

$$y^{(4)} - 2y'' + y = f, \tag{10}$$

$$y \in L_2(0,\infty), \tag{11}$$

$$y''(0) = y'''(0) = 0. \tag{12}$$

As we show below, for any function $f \in L_2(0,\infty)$ this problem has a unique solution y. The operator T is defined by the equality

$$Tf = y' - y''', \quad f \in L_2(0,\infty). \tag{13}$$

2. Properties of operator (10)–(13). In what follows we denote operator (10)–(13) by the symbol T. In two lemmas given below we prove that this operator is defined on the whole space $L_2(0,\infty)$, calculate its norm and deviation (1). We prove the lemmas using methods proposed by G. H. Hardy, J. E. Littlewood, and G. Pólya ([6], Chap. VII, § 7.8), which were further developed in [11]. Note that the same reasoning was used in papers [12] and [13].

Lemma 1. *For any function $f \in L_2(0,\infty)$ problem* (10)–(12) *has a unique solution, formula* (13) *defines a linear bounded operator in the space $L_2(0,\infty)$, and*

$$\|T\|_{L_2\to L_2} = \sqrt{\frac{4}{7}}.$$

Proof. The general solution to problem (10) is the sum of a general solution to the corresponding homogeneous equation $y^{(4)} - 2y'' + y = 0$ and a partial solution to the heterogeneous equation (10). The general solution to the homogeneous equation takes the form

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{x} + C_4 x e^{x}.$$

From condition (11) it follows that $C_3 = C_4 = 0$. Therefore, the function $y(x) = C_1 e^{-x} + C_2 x e^{-x}$ is a solution to the homogeneous equation.

Let us now find a partial solution to the heterogeneous equation (10) in the class W_2^4. To this end, we extend f (in an even way) to the whole numerical axis; we denote the obtained function by the same symbol f or f_c. Applying the Fourier operator

$$\widehat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i t x}\,dx$$

to Eq. (10), we obtain the equation

$$(2\pi i t)^4 \widehat{y}(t) - 2(2\pi i t)^2 \widehat{y}(t) + \widehat{y}(t) = \widehat{f}(t),$$

whence we get

$$\widehat{y}(t) = \frac{\widehat{f}(t)}{1 + 8\pi^2 t^2 + 16\pi^4 t^4}.$$

Evidently, the function $\widehat{y}$ belongs to the space $L_2(-\infty;\infty)$. Therefore, the function

$$y_p(x) = \int_{-\infty}^{\infty} \frac{\widehat{f}(t)}{1 + 8\pi^2 t^2 + 16\pi^4 t^4} e^{2\pi i x t}\,dt$$

also belongs to the space $L_2(-\infty;\infty)$ and represents a partial solution to the heterogeneous equation (10). Denote $h(t) = \frac{\widehat{f}_c(t)}{1+8\pi^2t^2+16\pi^4t^4}$ for $-\infty < t < \infty$. Due to the evenness of the function f we have the equality

$$y_p(x) = 2\int_0^{\infty} h(t)\cos(2\pi x t)\,dt.$$

Therefore the function y_p is real-valued and satisfies (10).

Thus, the solution to problem (10), (11) takes the form

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + 2\int_0^\infty h(t)\cos(2\pi xt)\,dt. \tag{14}$$

Assume that function (14) satisfies boundary conditions (12). Hence for coefficients C_1 and C_2 we obtain the following system of two linear equations:

$$\begin{aligned} C_1 - 2C_2 &= 8\pi^2\int_0^\infty h(t)t^2\,dt,\\ -C_1 + 3C_2 &= 0. \end{aligned}$$

The latter system has a unique solution. As a result, we obtain that the function

$$y(x) = 2\int_0^\infty h(t)\cos(2\pi xt)\,dt + e^{-x}(3+x)8\pi^2\int_0^\infty h(t)t^2\,dt$$

is a solution to problem (10)–(12).

Finally, the operator $Tf = y' - y'''$ takes the form:

$$Tf = 4\pi\int_0^\infty h(t)(4\pi^2t^2-1)t\sin(2\pi xt)\,dt - 16\pi^2 e^{-x}\int_0^\infty h(t)t^2\,dt. \tag{15}$$

For calculating the norm of the operator T we use the method proposed by G. H. Hardy, J. E. Littlewood, G. Pólya, and A. P. Buslaev. Consider the functional

$$J(y) = \int_0^\infty \big((y^{(4)} - 2y'' + y)^2 - \lambda^2(y' - y''')^2\big)dx = \|f\|^2_{L_2(0,\infty)} - \lambda^2\|Tf\|^2_{L_2(0,\infty)}$$

on the set of functions $y \in W_2^4$ such that $y''(0) = y'''(0) = 0$. Let us find the greatest value of λ, with which this functional is nonnegative. To this end, we define the nonnegative functional

$$K(y) = \int_0^\infty \big(y^{(4)} + Ay''' + By'' + Cy' + y\big)^2 dx.$$

Calculating integrals $J(y)$ and $K(y)$ by parts and taking into account that $y''(0) = y'''(0) = 0$, we obtain representations

$$\begin{aligned} K(y) = &\int_0^\infty \big((y^{(4)})^2 + (A^2-2B)(y''')^2 + (B^2-2AC+2)(y'')^2 + (C^2-2B)(y')^2 + y^2\big)dx\\ &- \big((BC-A)(y'(0))^2 + C(y(0))^2 y'(0)y(0)\big),\\ J(y) = &\int_0^\infty \big((y^{(4)})^2 + (4-\lambda^2)(y''')^2 + (6-2\lambda^2)(y'')^2 + (4-\lambda^2)(y')^2 + y^2\big)dx + 4y'(0)y(0). \end{aligned}$$

If coefficients A, B, and C satisfy the system

$$\begin{aligned} A^2 - 2B &= 4 - \lambda^2,\\ B^2 - 2AC &= 4 - 2\lambda^2,\\ C^2 - 2B &= 4 - \lambda^2, \end{aligned} \tag{16}$$

then the correlation $J(y) = K(y) + L(y)$ is fulfilled, where

$$L(y) = (BC-A)(y'(0))^2 + C(y(0))^2 + 2(B+2)y'(0)y(0).$$

If the quadratic form $L(y)$ of variables $y'(0)$ and $y(0)$ is non-negative definite, then the functional $J(y)$ is non-negative.

Consider a solution to system (16) in the form $A = C = \alpha$, $(B+2)^2 = 4\alpha^2$, where α is a real number (the second possible case $A = -C$ gives only the trivial solution $B = -2$, $A = C = \lambda = 0$). For the considered solution to the system, coefficients A, B, and C must be positive, otherwise the quadratic form $L(y)$ is not be positive definite, and $A = C = \alpha$, $B = 2\alpha - 2$, where $\alpha > 1$.

Now, if $(BC - A)C - (B + 2)^2 \geq 0$, then $L(y)$ is non-negative definite (the expression $BC - A$ is positive, because $C = \alpha > 1$). Substituting the solution to system (16) in this inequality, we get $\alpha^2(2\alpha - 7) \geq 0$. Then $\alpha \geq \frac{7}{2}$. As a result, the greatest value of λ such that $J(y) \geq 0$ is attained with $\alpha = \frac{7}{2}$ and equals $\sqrt{\frac{7}{4}}$.

Therefore,

$$\|T\| = \sup\left\{\frac{\|y' - y'''\|}{\|y^{(4)} - 2y'' + y\|} : y \in W_2^4,\ y \neq 0,\ y''(0) = y'''(0) = 0\right\} \leq \sqrt{\frac{7}{4}}.$$

With $\alpha = \frac{7}{2}$ functionals $K(y)$ and $L(y)$ take the form

$$K(y) = \int_0^\infty \left(y^{(4)} + \frac{7}{2}y''' + 5y'' + \frac{7}{2}y' + y\right)^2 dx,$$
$$L(y) = \frac{7}{2}(2y'(0) + y(0))^2.$$

According to the proof of Lemma 1 and Definition (13) of the operator T, if a function y is a solution to the equation

$$y^{(4)} + \frac{7}{2}y''' + 5y'' + \frac{7}{2}y' + y = 0 \tag{17}$$

and satisfies conditions

$$y(0) + 2y'(0) = 0, \quad y''(0) = y'''(0) = 0, \tag{18}$$

then it turns functionals $K(y)$ and $L(y)$ to zero. The general solution to Eq. (17) takes the form

$$y = C_1e^{-x} + C_2xe^{-x} + e^{-\frac{3}{4}x}\left(C_3\sin\frac{\sqrt{7}x}{4} + C_4\cos\frac{\sqrt{7}x}{4}\right);$$

here C_1, C_2, C_3, and C_4 are arbitrary real constants. Taking into account boundary conditions (18), finally we obtain

$$y(x) = C\left(-\sqrt{7}xe^{-x} + e^{-\frac{3}{4}x}\left(6\sin\frac{\sqrt{7}x}{4} + 2\sqrt{7}\cos\frac{\sqrt{7}x}{4}\right)\right), \quad C \in \mathbb{R}.$$

Hence it follows that the operator T attains its norm on functions

$$f(x) = y^{(4)}(x) - 2y''(x) + y(x) = C \cdot e^{-\frac{3}{4}x}\left(3\sin\frac{\sqrt{7}x}{4} - \sqrt{7}\cos\frac{\sqrt{7}x}{4}\right)$$

with $C \in \mathbb{R}$, $C \neq 0$. □

Let us find the value of the deviation

$$U(T) = \sup\{\|f' - Tf\|_{L_2(0,\infty)} : f \in W_2^2(0,\infty),\ \|f''\| \leq 1\}$$

for the operator T defined by formulas (13), (15).

Lemma 2. *The operator T defined by formulas* (13), (15) *satisfies the equality*

$$U(T) = \sqrt{\frac{4}{7}}. \tag{19}$$

Proof. Since $f = y^{(4)} - 2y'' + y$ and $Tf = y' - y'''$, we have

$$f' - Tf = y^{(5)} - y''', \quad f'' = y^{(6)} - 2y^{(4)} + y''.$$

Therefore,

$$U(T) = \sup\left\{ \frac{\|y^{(5)} - y'''\|}{\|y^{(6)} - 2y^{(4)} + y''\|} : y \in W_2^6,\ y^{(6)} - 2y^{(4)} + y'' \neq 0,\ y''(0) = y'''(0) = 0 \right\}. \tag{20}$$

The further reasoning is completely analogous to the proof of Lemma 1. Denote $z = y''$. Along with $U(T)$, we consider the value

$$\widetilde{U}(T) = \sup\left\{ \frac{\|z''' - z'\|}{\|z^{(4)} - 2z'' + z\|} : z \in W_2^4,\ z^{(4)} - 2z'' + z \neq 0,\ z(0) = z'(0) = 0 \right\}.$$

Evidently, $U(T) \le \widetilde{U}(T)$.

On the set of functions $z \in W_2^4$ such that $z(0) = z'(0) = 0$ we define the functional

$$J_1(z) = \int_0^\infty \big((z^{(4)} - 2z'' + z)^2 - \lambda^2 (z' - z''')^2\big)dx = \|f''\|^2_{L_2(0,\infty)} - \lambda^2 \|f' - Tf\|^2_{L_2(0,\infty)}.$$

We are interested in the greatest value of λ, with which this functional is non-negative.

Introduce a non-negative functional in the form

$$K_1(z) = \int_0^\infty \big(z^{(4)} + Az''' + Bz'' + Cz' + z\big)^2 dx.$$

As in the proof of the previous lemma, we ascertain that if coefficients A, B, and C satisfy the system

$$\begin{gathered} A^2 - 2B = 4 - \lambda^2, \\ B^2 - 2AC = 4 - 2\lambda^2, \\ C^2 - 2B = 4 - \lambda^2, \end{gathered}$$

then the equality $J_1(z) = K_1(z) + L_1(z)$ is fulfilled, where

$$L_1(z) = (AB - C)(z''(0))^2 + A(z'''(0))^2 + 2(B + 2)z''(0)z'''(0).$$

Therefore, the problem is symmetric to the problem in Lemma 1, and greatest λ, with which $L_1(z)$ is non-negative definite, again equals $\sqrt{\frac{7}{4}}$. Then $A = C = \frac{7}{2}$, $B = 5$, and

$$K_1(z) = \int_0^\infty \left(z^{(4)} + \frac{7}{2}z''' + 5z'' + \frac{7}{2}z' + z \right)^2 dx,$$
$$L_1(z) = \frac{7}{2}(2z''(0) + z'''(0))^2.$$

Therefore, $\widetilde{U}(T) \le \sqrt{\frac{4}{7}}$.

It is easy to see that if a function z is a solution to the equation

$$z^{(4)} + \frac{7}{2}z''' + 5z'' + \frac{7}{2}z' + z = 0 \tag{21}$$

and satisfies conditions $z'''(0) + 2z''(0) = 0$ and $z(0) = z'(0) = 0$, then it turns the value $\widetilde{U}(T)$ in expression (20) to its maximal value. By solving Eq. (21) subject to these boundary conditions we obtain functions

$$z(x) = c\bigg(-(7x + 21)e^{-x} + e^{-\frac{3}{4}x}\bigg(\sqrt{7}\sin\frac{\sqrt{7}x}{4} + 21\cos\frac{\sqrt{7}x}{4}\bigg)\bigg) \quad c \in \mathbb{R}, \tag{22}$$

which with $c \neq 0$ turn $\widetilde{U}(T)$ to its maximal value. Hence we get the equality

$$\widetilde{U}(T) = \sqrt{\frac{4}{7}}.$$

The maximum in (20) is attained on functions $y \in W_2^6$ which are connected with those (22) by the correlation $y'' = z$. We can easily make sure that

$$y(x) = C\left((-7x - 35)e^{-x} - \frac{1}{4}e^{-\frac{3}{4}x}\left(31\sqrt{7}\sin\frac{\sqrt{7}x}{4} - 21\cos\frac{\sqrt{7}x}{4}\right)\right), \quad C \in \mathbb{R}.$$

Therefore, $U(T) = \widetilde{U}(T) = \sqrt{\frac{4}{7}}$ and deviation (19) of the operator T is attained on functions

$$f(x) = y^{(4)}(x) - 2y''(x) + y(x) = C \cdot e^{-\frac{3}{4}x}\left(\sqrt{7}\sin\frac{\sqrt{7}x}{4} + 21\cos\frac{\sqrt{7}x}{4}\right)$$

with $C \in \mathbb{R}$, $C \neq 0$. □

3. The proof of the theorem. In view of Lemmas 1 and 2 we have

$$E\left(\sqrt{\frac{4}{7}}\right) \leq \sqrt{\frac{4}{7}}.$$

This is a particular case of inequality (9) with $N = \sqrt{\frac{4}{7}}$. The transition to arbitrary $N > 0$ can be realized by the well-known method [1]. Assume that S is a linear bounded operator in the space $L_2(0,\infty)$, for which the value of deviation (1) is finite. Using the operator S with $h > 0$, we construct an operator S_h by the following rule. We associate a function $f \in L_2$ and a parameter $h > 0$ with the function $f_h(x) = \frac{1}{h}f(hx)$, $x \in (0,\infty)$. Now we define the operator S_h by the formula

$$(S_h f)(x) = (S f_h)\left(\frac{x}{h}\right), \quad f \in L_2. \tag{23}$$

It is not difficult to verify [1] that the following two correlations are valid:

$$\|S_h\| = \frac{\|S\|}{h}, \quad U(S_h) = hU(S).$$

In particular, for the operator T defined in (10)–(12) formula (23) gives the operator T_h such thst

$$\|T_h\| = h^{-1}\sqrt{\frac{4}{7}}, \quad U(T_h) = h\sqrt{\frac{4}{7}}.$$

For $N > 0$ we choose a parameter $h = h(N) > 0$ so that

$$\|T_h\| = h^{-1}\sqrt{\frac{4}{7}} = N,$$

i.e., we put $h = h(N) = N^{-1}\sqrt{\frac{4}{7}}$. As a result, we obtain

$$U(N) \leq U(T_{h(N)}) = \frac{4}{7N}.$$

Thus, the theorem is proved.

4. The optimal differentiation of functions given with an error. Problem (2) is connected not only with the exact inequality (3), but also with the optimal differentiation of functions (or the optimal restoration of a differentiation operator on functions) from the space $W_2^2(0,\infty)$ which are defined with some error in $L_2(0,\infty)$ (e.g., [1–3, 14]). Let $\mathscr{R}$ be one of the following three sets of mappings from $L_2(0,\infty)$ to $L_2(0,\infty)$: the set $\mathscr{B} = \mathscr{B}_2(0,\infty)$ of all linear bounded operators, the set $\mathscr{L} = \mathscr{L}_2(0,\infty)$ of all linear operators, and the set $\mathscr{O} = \mathscr{O}_2(0,\infty)$ of all one-valued mappings.

For an operator $S \in \mathscr{R}$ and a number $\delta \geq 0$ we set

$$U_\delta(S) = \sup\{\|f' - S\eta\| : f \in Q_2^2,\ \eta \in L_2(0,\infty),\ \|f - \eta\| \leq \delta\}.$$

Then

$$\mathcal{E}_\delta(\mathscr{R}) = \inf\{U_\delta(S) :\ S \in \mathscr{R}\} \tag{24}$$

is the value of the error of the optimal differentiation of functions from the class $Q_2^2(0,\infty)$ which are defined with some error δ with the help of the set of restoration methods $\mathscr{R}$. In view of inclusions $\mathscr{B} \subset \mathscr{L} \subset \mathscr{O}$ we have inequalities

$$\mathcal{E}_\delta(\mathscr{O}) \leq \mathcal{E}_\delta(\mathscr{L}) \leq \mathcal{E}_\delta(\mathscr{B}).$$

As a particular case of general results obtained by S. B. Stechkin and his followers, we get inequalities (e.g., [3], items 2.1, 4.1, 4.4):

$$\sqrt{2\delta} \leq \mathcal{E}_\delta(\mathscr{R}) \leq \inf\{E(N) + N\delta : N \geq 0\}, \quad \delta > 0.$$

By applying estimate (9) we deduce the following corollary.

Corollary. For the optimal restoration value (24) with the help of each of three sets of methods $\mathscr{B}$, $\mathscr{L}$, and $\mathscr{O}$ we obtain estimates

$$\sqrt{2\delta} \leq \mathcal{E}_\delta(\mathscr{R}) \leq 4\sqrt{\frac{\delta}{7}}, \quad \delta > 0.$$

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REFERENCES

1. S. B. Stechkin, "Best Approximation of Linear Operators," Matem. Zametki **1** (6), 137–148 (1967).
2. V. V. Arestov and V. N. Gabushin, "Best Approximation of Unbounded Operators by Bounded Ones," Izv. Vyssh. Uchebn. Zaved. Mat., No. 11, 42–68 (1995) [Russian Mathematics (Iz. VUZ) **39** (11), 38–63 (1995)].
3. V. V. Arestov, "Approximation of Unbounded Operators by Bounded Operators and Related Extremal Problems," Uspekhi Mat. Nauk **51** (6), 89–124 (1996).
4. V. F. Babenko, N. P. Korneichuk, V. A. Kofanov, and S. A. Pichugov, *Inequalities for Derivatives and Their Applications* (Naukova Dumka, Kiev, 2003) [in Russian].
5. Yu. N. Subbotin and L. V. Taikov, "Best Approximation of a Differentiation Operator in L_2-Space," Matem. Zametki **3** (2), 157–164 (1968).
6. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge University Press, Cambridge, 1934; Inost. Lit., Moscow, 1948).
7. T. Kato, "On an Inequality of Hardy, Littlewood, and Pólya," Adv. Math. **7** (3), 217–218 (1971).
8. W. D. Everitt and A. Zettl, "On a Class of Integral Inequalities," J. London Math. Soc. **17**, 291–303 (1978).
9. M. K. Kwong and A. Zettl, *Norm Inequalities for Derivatives and Differences* (Lecture Notes in Mathematics 1536, Springer-Verlag, Berlin, 1992).
10. N. P. Kuptsov, "Kolmogorov Estimates for Derivatives in $L_2[0,\infty)$," Trudy MIAN SSSR **138**, 94–117 (1975).
11. A. P. Buslaev, "An Extremal Problem Connected with Inequalities for Derivatives," Vestn. Mosk. Gos. Univ. Ser. Matem., Mekhan., No. 3, 67–77 (1978).
12. A. L. Rublev, "On Approximation of the Differentiation Operator over the Class of Twice Differentiable Functions in the Space $L_2(0,\infty)$," Trudy Inst. Mat. i Mekh. UrO RAN, **4**, 162–170 (1996).
13. E. E. Berdysheva, "On the Best Approximation of the Differentiation Operator in $L_2(0,\infty)$," East J. Approx. **2** (3), 281–287 (1996).
14. V. K. Ivanov, V. V. Vasin, and V. P. Tanana, *Theory of Linear Ill-posed Problems and Its Applications* (Nauka, Moscow, 1978) [in Russian].

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